GLOBAL WELL-POSEDNESS OF KIRCHHOFF SYSTEMS

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ABSTRACT. The aim of this paper is to establish the H^1 global well-posedness for Kirchhoff systems. The new approach to the construction of solutions is based on the asymptotic integrations for strictly hyperbolic systems with time-dependent coefficients. These integrations play an important rôle to setting the subsequent fixed point argument. The existence of solutions for less regular data is discussed, and several examples and applications are presented.

1. INTRODUCTION

The Kirchhoff equations of the form

$$(1.1) \qquad \partial_t^2 u - a\left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right) \Delta u = 0 \quad (t \in \mathbb{R},\, x \in \mathbb{R}^n)$$

have been previously considered for various positive functions $a(s)$. Bernstein first studied the global existence for real analytic data (see [1]), and after him, many authors investigated these equations further (see [4, 5, 6, 8, 11, 19, 21, 23]). Also, the global existence for quasi-analytic data was studied by Nishihara (see [20]), and variants of the class in [20] were discussed in [7, 10, 13].

The approach of this paper yields new results already in the scalar case of the classical Kirchhoff equation (1.1) but, in fact, we are able to make advances for coupled equations as well or, more generally, for Kirchhoff systems. To this end, Kirchhoff systems are of interest but present several major complications compared to the scalar case. First of all, even for the linearised system, it is much more difficult to find a suitable representation of solutions which would, on one hand, work with the low regularity (C^1) of coefficients while, on the other hand, allow one to obtain sufficiently good estimates for solutions. Moreover, in the case of systems of higher order, it is impossible to find its characteristics explicitly, and the geometry of the system or rather of the level sets of the characteristics enters the picture.

The main new idea (even for the classical equation (1.1)) behind this paper is to approach the problem by developing the "asymptotic integration" method for the linearised equation to be able to control its solutions to the extent of being able to prove a-priori estimates necessary for the handling of the fully nonlinear problem. Thus, for the linear strictly hyperbolic systems we developed in [16] the method of

asymptotic integrations allowing us to obtain representations of solutions under the low regularity of the coefficients. Consequently, we apply it in the present setting to set up a suitable fixed point argument assuring the well-posedness of the Cauchy problem. The results presented in this paper resolve the well-posedness problem for a general class of strictly hyperbolic systems. Moreover, even for the classical Kirchhoff-type equation (1.1) we obtain new results. Thus, the regularity of data in Theorem 2.7 is lower than that in [2, 4]. Moreover, we prove the well-posedness in low dimensions $n = 1, 2$ which remained open since D'Ancona and Spagnolo [4].

We consider the Kirchhoff-type systems of the form

$$(1.2) \qquad \begin{cases} D_t U = A(s(t), D_x)U, \quad t \neq 0, \quad x \in \mathbb{R}^n, \\ U(0, x) = U_0(x) = {}^T\left(f_0(x), f_1(x), \ldots, f_{m-1}(x)\right), \quad x \in \mathbb{R}^n, \end{cases}$$

where $D = -i\partial$ and $A(s, D_x)$ is a first order $m \times m$ pseudo-differential system with a suitably smooth behaviour in $s \in \mathbb{R}$ in a neighbourhood of 0; $s(t)$ is a quadratic form defined to be

$$(1.3) \qquad s(t) = \langle SU(t, \cdot), U(t, \cdot)\rangle_{\mathbb{L}^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} {}^T(SU(t, x))\overline{U(t, x)}\, dx$$

for some $m \times m$ Hermitian matrix S, and we put $\mathbb{L}^2(\mathbb{R}^n) = (L^2(\mathbb{R}^n))^m$. We allow the operator $A(s, D_x)$ to be pseudo-differential since we want our analysis to be applicable to scalar higher order equations and to coupled equations of higher orders e.g. to coupled Kirchhoff equations, see Example 2.2 and Example 2.4. The precise meaning of "pseudo-differential" in this context will be specified below.

We assume that the system (1.2) is strictly hyperbolic. Namely, the characteristic polynomial of the differential operator $D_t - A(s, D_x)$ has real and distinct roots $\varphi_1(s, \xi), \ldots, \varphi_m(s, \xi)$ for any s in the domain of the definition of matrices $A(s, \xi)$ and for any $\xi \in \mathbb{R}^n \backslash 0$, i.e.

$$(1.4) \qquad \det(\tau I - A(s, \xi)) = (\tau - \varphi_1(s, \xi)) \cdots (\tau - \varphi_m(s, \xi)).$$

We assume that $A(s, \xi)$ is positively homogeneous in ξ of order one, i.e. we have $A(s, \lambda\xi) = \lambda A(s, \xi)$ for all $s \in [0, \delta]$ for a suitable (usually sufficiently small) $\delta > 0$, $\xi \neq 0$, and $\lambda > 0$. Then by the strict hyperbolicity, we may assume that

$$(1.5) \qquad \inf_{s \in [0, \delta],\, |\xi| = 1} |\varphi_j(s, \xi) - \varphi_k(s, \xi)| > 0 \quad \text{for } j \neq k.$$

In the last part of the next section some examples of (1.2) will be presented (see Examples 2.2, 2.4 and 2.6).

One of the main difficulties in the analysis of Kirchhoff equations is that even if the natural H^1 well-posedness of (1.2) holds, this would mean that the function $s(t)$ is at most C^1. Consequently, even in the case of a linear system (1.2) we would need to analyse systems with low regularity C^1 of the coefficients, in which case dispersive and Strichartz estimates are more difficult to obtain due to the lack of a satisfactory representation for solutions. Such analysis with applications to nonlinear perturbations and scattering for systems (1.2) will appear elsewhere.

The global existence of (1.2) for differential systems was analysed by Callegari & Manfrin (see [2], and also [12]), where the Cauchy data are either smooth compactly supported or belong to a certain special class $\mathscr{M}(\mathbb{R}^n)$, which contains a weighted

Sobolev space, and the system is C^2 in time. Precise definition of this class will be introduced as a remark after the statement of Theorem 2.1. The purpose in the present paper is to find global solutions to (1.2) for a more general class of data, removing the smooth compactly supported data assumption in the general result. Our approach is to employ asymptotic integrations for *linear* hyperbolic systems with time-dependent coefficients in order to derive a certain integrability of the time-derivative of coefficients (see Lemma 4.1 in §4), which enables us to use the fixed point argument. It should be noted that the present argument will also resolve an open problem of the well-posedness in low dimensions in D'Ancona & Spagnolo [4] (see Theorem 2.7).

This paper is organised as follows. In §2 we formulate the main result and give examples and several corollaries for equations of different types. In §3 we will introduce asymptotic integrations for linear hyperbolic systems with time-dependent coefficients, which enable us to prove Theorem 2.1. The proof of Theorem 2.1 will be given in §4.

2. GLOBAL WELL-POSEDNESS FOR KIRCHHOFF EQUATIONS AND SYSTEMS

To state the main result, let us introduce a class of data which ensures the global well-posedness for (1.2). A class $\mathscr{Y}(\mathbb{R}^n)$ consists of all $U_0 = {}^T(f_0, f_1, \ldots, f_{m-1}) \in (\mathscr{S}'(\mathbb{R}^n))^m$ such that

$$|U_0|_{\mathscr{Y}(\mathbb{R}^n)} := \sum_{j,k=0}^{m-1} \int_{-\infty}^{\infty} \left(\int_{\mathbb{S}^{n-1}} \left| \int_0^\infty e^{i\tau\rho} \widehat{f_j}(\rho\omega) \overline{\widehat{f_k}(\rho\omega)} \rho^n \, d\rho \right| d\sigma(\omega) \right) d\tau < \infty,$$

where $\mathscr{S}'(\mathbb{R}^n)$ is the space of tempered distributions on $\mathbb{R}^n$, and $\mathbb{S}^{n-1}$ is $(n-1)$-dimensional sphere and $d\sigma(\omega)$ is the $(n-1)$-dimensional Hausdorff measure.

We denote $\mathbb{H}^\sigma(\mathbb{R}^n) = (H^\sigma(\mathbb{R}^n))^m$ for $\sigma \in \mathbb{R}$, where $H^\sigma(\mathbb{R}^n) = \langle D \rangle^{-\sigma} L^2(\mathbb{R}^n)$ are the standard Sobolev spaces, and $\langle D \rangle = (1 - \Delta)^{1/2}$. The space $\mathbb{H}^\sigma_\varkappa(\mathbb{R}^n)$ denotes the m direct product of weighted Sobolev spaces $H^\sigma_\varkappa(\mathbb{R}^n)$, which consist of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $\langle x \rangle^\varkappa f \in H^\sigma(\mathbb{R}^n)$, and $\langle x \rangle = (1 + |x|^2)^{1/2}$. Then by using Lemma A.1 in [5], we conclude that

$$(2.6) \qquad \mathbb{H}^1_\varkappa(\mathbb{R}^n) \subset \mathscr{Y}(\mathbb{R}^n), \quad \forall \varkappa > 1.$$

For a function, say $\varphi(\xi)$, positively homogeneous of order one in ξ, we can factor out ξ and restrict the function φ to the unit sphere; in this case we will be using the notations like $\varphi(\xi/|\xi|) \in L^\infty(\mathbb{R}^n \backslash 0)$ instead of $L^\infty(\mathbb{R}^n)$, since an extension of $\varphi(\xi/|\xi|)$ to $\xi = 0$ is irrelevant for our analysis.

We shall prove here the following:

Theorem 2.1. *Assume that system* (1.2) *is strictly hyperbolic, and that* $A(s, \xi) = (a_{jk}(s, \xi))_{j,k=1}^m$ *is an* $m \times m$ *matrix, positively homogeneous of order one in* ξ, *whose entries* $a_{jk}(s, \xi/|\xi|)$ *are in* $\mathrm{Lip}([0, \delta]; L^\infty(\mathbb{R}^n \backslash 0))$ *for some* $\delta > 0$. *If* $U_0 \in \mathbb{L}^2(\mathbb{R}^n) \cap \mathscr{Y}(\mathbb{R}^n)$ *satisfies*

$$(2.7) \qquad \|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)}^2 + |U_0|_{\mathscr{Y}(\mathbb{R}^n)} \ll 1,$$

then system (1.2)-(1.3) *has a solution* $U(t,x) \in C(\mathbb{R}; \mathbb{L}^2(\mathbb{R}^n))$. *In addition to* (2.7), *if* $U_0 \in \mathbb{H}^1(\mathbb{R}^n)$, *then the solution* $U(t,x)$ *exists uniquely in the class* $C(\mathbb{R}; \mathbb{H}^1(\mathbb{R}^n)) \cap C^1(\mathbb{R}; \mathbb{L}^2(\mathbb{R}^n))$.

As a related result to Theorem 2.1, Callegari & Manfrin introduced the following class (see [2]):

$$\mathscr{M}(\mathbb{R}^n) = \left\{ U_0(x) = {}^T(f_0(x), f_1(x), \ldots, f_{m-1}(x)) \in (\mathscr{S}'(\mathbb{R}^n))^m : |U_0|_{\mathscr{M}(\mathbb{R}^n)} < \infty \right\},$$

where

$$|U_0|_{\mathscr{M}(\mathbb{R}^n)} = \sum_{k=0}^{2} \sum_{j=0}^{m-1} \sup_{\omega \in \mathbb{S}^{n-1}} \int_0^\infty \left| \partial_\rho^k \widehat{f_j}(\rho\omega) \right|^2 \left(1 + \rho^{\max\{n,2\}} \right) d\rho.$$

Hence, in particular, Theorem 2.1 generalises [2] since the inclusion among this class and ours is:

$$(C_0^\infty(\mathbb{R}^n))^m \subset \mathbb{L}_2^1(\mathbb{R}^n) \cap \mathbb{H}_2^1(\mathbb{R}^n) \subset \mathscr{M}(\mathbb{R}^n) \subset \mathscr{Y}(\mathbb{R}^n),$$

where $\mathbb{L}_2^1(\mathbb{R}^n)$ is the m direct product of $L_2^1(\mathbb{R}^n) = \{ f \in \mathscr{S}'(\mathbb{R}^n) : \langle x \rangle^2 f \in L^1(\mathbb{R}^n) \}$.

Needless to say, Theorem 2.1 covers the second order case, i.e., the Kirchhoff equation

$$\partial_t^2 u - \left(1 + \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \right) \Delta u = 0.$$

In this case, Yamazaki found a general class that ensures global well-posedness (see [23]). In fact, she proved that the space $H_\varkappa^2(\mathbb{R}^n) \times H_\varkappa^1(\mathbb{R}^n)$ introduced by D'Ancona & Spagnolo (see [5]) is contained in $\mathscr{Y}_\varkappa(\mathbb{R}^n)$ for any $\varkappa \in (1, n+1]$. The class $\mathscr{Y}_\varkappa(\mathbb{R}^n)$ consists of the pairs of data $(f_0, f_1) \in H^{3/2}(\mathbb{R}^n) \times H^{1/2}(\mathbb{R}^n)$ such that

$$\sum_{j,k=0}^{1} \sup_{\tau \in \mathbb{R}} \langle \tau \rangle^\varkappa \left| \int_{\mathbb{R}^n} e^{i\tau|\xi|} \widehat{f_j}(\xi) \overline{\widehat{f_k}(\xi)} |\xi|^{3-j-k} \, d\xi \right| < \infty.$$

After her, Kajitani found the most general class $\mathscr{K}(\mathbb{R}^n)$:

$$\mathscr{K}(\mathbb{R}^n) = \left\{ (f_0, f_1) \in H^{3/2}(\mathbb{R}^n) \times H^{1/2}(\mathbb{R}^n) : |(f_0, f_1)|_{\mathscr{K}(\mathbb{R}^n)} < \infty \right\},$$

where

$$|(f_0, f_1)|_{\mathscr{K}(\mathbb{R}^n)} = \sum_{j,k=0}^{1} \int_{-\infty}^{\infty} \left| \int_{\mathbb{R}^n} e^{i\tau|\xi|} \widehat{f_j}(\xi) \overline{\widehat{f_k}(\xi)} |\xi|^{3-j-k} \, d\xi \right| d\tau$$

(see [11], and also Rzymowski [21] who considered the one-dimensional case). As to the exterior version of the class $\mathscr{K}(\mathbb{R}^n)$, we can refer to the recent results [14, 15] (see also [24, 25]). The inclusions among these classes and ours are:

$$H_\varkappa^2(\mathbb{R}^n) \times H_\varkappa^1(\mathbb{R}^n) \subset \begin{cases} \mathscr{Y}_\varkappa(\mathbb{R}^n) \\ \mathscr{Y}(\mathbb{R}^n) \end{cases} \subset \mathscr{K}(\mathbb{R}^n).$$

Here the first inclusion holds for $\varkappa \in (1, n+1]$ and the second one is valid for any $\varkappa > 1$.

In the rest of this section, let us give some examples of applications of our result. First of all, we note that Theorem 2.1 covers all the examples of Callegari and Manfrin

[2], in particular, the Kirchhoff equations of higher order etc. There, it is assumed that the Cauchy data $f_k(x)$, $k = 0, 1, \ldots, m - 1$, belong to $\mathscr{M}(\mathbb{R}^n)$, or even $C_0^\infty(\mathbb{R}^n)$. More precisely, we have:

Example 2.2. *Let us consider the Cauchy problem*

(2.8)
$$
\begin{cases}
L(D_t, D_x, s(t))\, u \equiv D_t^m u + \sum_{\substack{|\nu|+j=m \\ j \le m-1}} b_{\nu,j}(s(t))\, D_x^\nu D_t^j u = 0, \\[2mm]
D_t^k u(0, x) = f_k(x), \quad k = 0, 1, \cdots, m-1.
\end{cases}
$$

Here the quadratic form $s(t)$ is given by

$$
s(t) = \int_{\mathbb{R}^n} \sum_{|\beta|=|\gamma|=m-1} s_{\beta\gamma} D^\beta u(t, x) \overline{D^\gamma u(t, x)}\, dx,
$$

where $\beta = (\beta_t, \beta_x)$, $\gamma = (\gamma_t, \gamma_x)$, $D^\beta = D_t^{\beta_t} D_x^{\beta_x}$ and $s_{\beta\gamma} = \overline{s_{\gamma\beta}}$. We assume that the symbol $L(\tau, \xi, s)$ of the differential operator $L(D_t, D_x, s)$ has real and distinct roots $\varphi_1(s, \xi), \ldots, \varphi_m(s, \xi)$ for $\xi \ne 0$ and $0 \le s \le \delta$ with $\delta > 0$, i.e.,

(2.9)
$$
L(\tau, \xi, s) = (\tau - \varphi_1(s, \xi)) \cdots (\tau - \varphi_m(s, \xi)),
$$

(2.10)
$$
\inf_{s \in [0,\delta], |\xi|=1} |\varphi_j(s, \xi) - \varphi_k(s, \xi)| > 0 \quad for\ j \ne k.
$$

By taking the Fourier transform in the space variables and introducing the vector

$$
V(t, \xi) = {}^T(|\xi|^{m-1}\widehat{u}(t, \xi), |\xi|^{m-2}D_t\widehat{u}(t, \xi), \cdots, D_t^{m-1}\widehat{u}(t, \xi)),
$$

we reduce the problem to the system

$$
D_t V = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
\vdots & \ddots & \ddots & 1 \\
-H_m(s(t), \xi) & -H_{m-1}(s(t), \xi) & \cdots & -H_1(s(t), \xi)
\end{pmatrix} |\xi| V
$$
$$
= A(s(t), \xi) V,
$$

where we put

$$
H_j(s(t), \xi) = \sum_{|\nu|=j} b_{\nu, m-j}(s(t))(\xi/|\xi|)^\nu, \quad (j = 1, \ldots, m).
$$

Then we have:

Theorem 2.3. *Assume (2.9)–(2.10). If $f_k \in H^{m-k}(\mathbb{R}^n)$ $(k = 0, 1, \ldots, m - 1)$, then (2.8) has a unique solution $u(t, x) \in \bigcap_{k=0}^{m-1} C^k(\mathbb{R}; H^{m-k}(\mathbb{R}^n))$, provided that the quantity*

$$
\|(|D_x|^{m-1}f_0, |D_x|^{m-2}f_1, \ldots, f_{m-1})\|_{\mathbb{L}^2(\mathbb{R}^n)}^2 + |(|D_x|^{m-1}f_0, |D_x|^{m-2}f_1, \ldots, f_{m-1})|_{\mathscr{Y}(\mathbb{R}^n)}
$$

is sufficiently small.

As a new example of (1.2), we can treat the completely coupled Kirchhoff equations of the following type.

Example 2.4. *Let us consider the Cauchy problem*

$$(2.11) \quad \begin{cases} \partial_t^2 u - a_1 \left(1 + \|\nabla u(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2\right) \Delta u + P_1(t, D_x)v = 0, \\ \partial_t^2 v - a_2 \left(1 + \|\nabla u(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2\right) \Delta v + P_2(t, D_x)u = 0, \\ \partial_t^j u(0, x) = u_j(x), \quad \partial_t^j v(0, x) = v_j(x), \quad j = 0, 1, \end{cases}$$

for some second order homogeneous polynomials $P_1(t, D_x), P_2(t, D_x)$, and for some constants $a_1, a_2 > 0$ with $a_1 \neq a_2$. The quadratic form is given here by

$$s(t) = \|\nabla u(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2.$$

We assume that

$$(2.12) \quad |\xi|^{-2} P_k(t, \xi) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}; L^\infty(\mathbb{R}^n \backslash 0)), \quad |\xi|^{-2} \partial_t P_k(t, \xi) \in L^1(\mathbb{R}; L^\infty(\mathbb{R}^n \backslash 0))$$

for $k = 1, 2$, and that

$$(2.13) \quad \inf_{t \in \mathbb{R}, |\xi|=1} \left\{ (a_1 - a_2)^2 + 4P_1(t, \xi)P_2(t, \xi) \right\} > 0,$$

$$(2.14) \quad \inf_{t \in \mathbb{R}, |\xi|=1} \left\{ a_1^2 a_2^2 - P_1(t, \xi)P_2(t, \xi) \right\} > 0.$$

By taking the Fourier transform in the space variables and introducing the vector

$$V(t, \xi) = {}^T(|\xi|\widehat{u}(t, \xi), \widehat{u}'(t, \xi), |\xi|\widehat{v}(t, \xi), \widehat{v}'(t, \xi)),$$

we rewrite (2.11) as a system

$$D_t V = \begin{pmatrix} 0 & -i|\xi| & 0 & 0 \\ ic_1(t)^2|\xi| & 0 & iP_1(t, \xi)|\xi|^{-1} & 0 \\ 0 & 0 & 0 & -i|\xi| \\ iP_2(t, \xi)|\xi|^{-1} & 0 & ic_2(t)^2|\xi| & 0 \end{pmatrix} V =: A(s(t), \xi)V,$$

where

$$c_k(t) = \sqrt{a_k(1 + s(t))}, \quad k = 1, 2.$$

The four characteristic roots of the equation

$$\det(\tau I - A(s(t), \xi)) = 0$$

in τ are given by

$$\varphi_{1,2,3,4}(s(t), \xi) =$$
$$\pm \frac{|\xi|}{\sqrt{2}} \sqrt{c_1(t)^2 + c_2(t)^2 \pm \sqrt{\{c_1(t)^2 - c_2(t)^2\}^2 + 4P_1(t, \xi)P_2(t, \xi)|\xi|^{-4}}}.$$

Then it follows from (2.12)–(2.14) that

$$\inf_{s \in [0, \delta], |\xi|=1} |\varphi_j(s, \xi) - \varphi_k(s, \xi)| > 0 \quad \text{for } j \neq k.$$

Thus we have the following:

Theorem 2.5. *Assume (2.12)–(2.14). If $(u_j, v_j) \in \mathbb{H}^{2-j}(\mathbb{R}^n)$ for $j = 0, 1$, then (2.11) has a pair of unique solutions $(u, v) \in \bigcap_{k=0,1} C^k(\mathbb{R}; \mathbb{H}^{2-k}(\mathbb{R}^n))$ provided that the quantity*

$$\|(|D_x|u_0, u_1, |D_x|v_0, v_1)\|_{\mathbb{L}^2(\mathbb{R}^n)}^2 + |(|D_x|u_0, u_1, |D_x|v_0, v_1)|_{\mathscr{Y}(\mathbb{R}^n)}$$

is sufficiently small.

Theorem 2.1 can be also generalised in another direction. In fact, as it is pointed out in [2], the nonlocal term (1.3) may be replaced by

$$s(t) = \left\langle |\xi|^{-k} S \widehat{U}(t,\xi), \widehat{U}(t,\xi) \right\rangle_{\mathbb{L}^2(\mathbb{R}^n)}$$

for $0 \leq k \leq n-1$. By this little change we can generalize Theorem 2.3 without any change in the proof. More precisely, we have the following example, which resolves an open problem in D'Ancona & Spagnolo [4].

Example 2.6. *Let us consider the Cauchy problem for the second order equation of the form*

$$(2.15) \qquad \partial_t^2 u - \left(1 + \int_{\mathbb{R}^n} |u(t,x)|^2\, dx\right) \Delta u = 0, \quad t \neq 0, \quad x \in \mathbb{R}^n,$$

with data

$$(2.16) \qquad\qquad u(0,x) = f_0(x), \quad \partial_t u(0,x) = f_1(x).$$

In this particular case, the nonlocal term $s(t)$ is defined by

$$s(t) = \|u(t)\|^2_{L^2(\mathbb{R}^n)}.$$

Introducing another class of data

$$\widetilde{\mathscr{Y}}(\mathbb{R}^n) = \left\{ (f_0, f_1) \in \mathscr{S}'(\mathbb{R}^n) \times \mathscr{S}'(\mathbb{R}^n) : |(f_0, f_1)|_{\widetilde{\mathscr{Y}}(\mathbb{R}^n)} < \infty \right\},$$

where we put

$$|(f_0, f_1)|_{\widetilde{\mathscr{Y}}(\mathbb{R}^n)} = \sum_{j,k=0}^{1} \int_{-\infty}^{\infty} \left(\int_{\mathbb{S}^{n-1}} \left| \int_0^{\infty} e^{i\tau\rho} \widehat{f_j}(\rho\omega)\overline{\widehat{f_k}(\rho\omega)} \rho^{n-j-k}\, d\rho \right| d\sigma(\omega) \right) d\tau,$$

we have:

Theorem 2.7. *Let $n \geq 1$. Then, for any $(f_0, f_1) \in (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)) \cap \widetilde{\mathscr{Y}}(\mathbb{R}^n)$, (2.15)–(2.16) has a unique solution $u \in \bigcap_{k=0,1} C^k(\mathbb{R}; H^{1-k}(\mathbb{R}^n))$, provided that the quantity*

$$\|f_0\|^2_{L^2(\mathbb{R}^n)} + \|f_1\|^2_{\dot{H}^{-1}(\mathbb{R}^n)} + |(f_0, f_1)|_{\widetilde{\mathscr{Y}}(\mathbb{R}^n)}$$

is sufficiently small. Here $\dot{H}^{-1}(\mathbb{R}^n) = |D_x| L^2(\mathbb{R}^n)$ is the homogeneous Sobolev space of order -1.

Let us give a few remarks on Theorem 2.7. This theorem generalises the results of [2] and [4]. Indeed, when the space dimension n is greater than 2, $n \geq 3$, a similar result was obtained in [2] and [4]. However, the regularity of data in Theorem 2.7 is lower than that in [2, 4]. It should be noted that Theorem 2.7 also covers low dimensions $n = 1, 2$, the case that remained open since [2, 4].

3. Asymptotic integrations for linear hyperbolic system

In this section we shall derive asymptotic integrations for linear hyperbolic systems with time-dependent coefficients, a kind of representation formula for their solutions. In fact, we have discussed such arguments in our recent paper [17] in the context of the scattering problems. To make the argument self-contained, we must give the proof completely, because the Fourier integral form of solutions U to Kirchhoff system (1.2) will be obtained by a careful observation of the construction of asymptotic integrations for linear systems. We note that the major advantage of the asymptotic integration method developed in [16] in comparison to other approaches, e.g. the diagonalisation schemes for systems as in [22], is the low C^1 regularity of coefficients in t sufficient for the construction compared to higher regularity required for other methods.

Let us consider the linear Cauchy problem

$$(3.17) \qquad \begin{cases} D_t U = A(t, D_x)U, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ U(0, x) = {}^T\left(f_0(x), f_1(x), \ldots, f_{m-1}(x)\right) \in (C_0^\infty(\mathbb{R}^n))^m, \end{cases}$$

where $A(t, D_x)$ is a first order $m \times m$ pseudo-differential system, with symbol $A(t, \xi)$ of the form $A(t, \xi) = (a_{ij}(t, \xi))_{i,j=1}^m$, positively homogeneous of order one in ξ. We assume that

$$(3.18) \quad a_{ij}(t, \xi/|\xi|) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}; L^\infty(\mathbb{R}^n \backslash 0)) \text{ and } \partial_t a_{ij}(t, \xi/|\xi|) \in L^1(\mathbb{R}; L^\infty(\mathbb{R}^n \backslash 0)),$$

and that system (3.17) is strictly hyperbolic:

$$(3.19) \qquad \det(\tau I - A(t, \xi)) = 0 \text{ has real and distinct roots } \varphi_1(t, \xi), \ldots, \varphi_m(t, \xi),$$

i.e.,

$$(3.20) \qquad \inf_{t \in \mathbb{R}, |\xi|=1} |\varphi_j(t, \xi) - \varphi_k(t, \xi)| > 0 \quad \text{for } j \neq k.$$

Notice that each characteristic root $\varphi_j(t, \xi)$ is positively homogeneous of order one in ξ.

Let us first analyse certain basic properties of characteristic roots $\varphi_k(t, \xi)$. The next proposition is established in [17].

Proposition 3.1 ([17] (Proposition 2.1)). *Let $D_t - A(t, D_x)$ be a strictly hyperbolic operator as above. If $a_{ij}(t, \xi/|\xi|)$ belong to $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}; L^\infty(\mathbb{R}^n \backslash 0))$ for $i, j = 1, \ldots, m$, then $|\varphi_k(t, \xi)| \leq C|\xi|$ for some $C > 0$, and functions $\partial_t \varphi_k(t, \xi)$, $k = 1, \ldots, m$, are positively homogeneous of order one in ξ. In addition, if $\partial_t a_{ij}(t, \xi/|\xi|)$ belong to $L^1(\mathbb{R}; L^\infty(\mathbb{R}^n \backslash 0))$ for $i, j = 1, \ldots, m$, then we have also*

$$\partial_t \varphi_k(t, \xi/|\xi|) \in L^1(\mathbb{R}; L^\infty(\mathbb{R}^n \backslash 0)).$$

Proof. Let us show first that $\varphi_k(t, \xi)$ are bounded with respect to $t \in \mathbb{R}$, i.e.,

$$(3.21) \qquad |\varphi_k(t, \xi)| \leq C|\xi|, \quad \text{for all } \xi \in \mathbb{R}^n, \, t \in \mathbb{R}, \, k = 1, \ldots, m.$$

We will use the fact that $\varphi_k(t, \xi)$ are roots of the polynomial

$$L(t, \tau, \xi) = \det(\tau I - A(t, \xi))$$

of the form

$$L(t, \tau, \xi) = \tau^m + \alpha_1(t, \xi)\tau^{m-1} + \cdots + \alpha_m(t, \xi)$$

with $|\alpha_j(t, \xi)| \leq M|\xi|^j$, for some $M \geq 1$, where

$$\alpha_j(t, \xi) = (-1)^j \sum_{i_1 < i_2 < \cdots < i_j} \det \begin{pmatrix} a_{i_1 i_1}(t, \xi) & \cdots & a_{i_1 i_j}(t, \xi) \\ \vdots & \ddots & \vdots \\ a_{i_j i_1}(t, \xi) & \cdots & a_{i_j i_j}(t, \xi) \end{pmatrix}.$$

Suppose that one of its roots τ satisfies $|\tau(t, \xi)| > 2M|\xi|$. Then

$$|L(t, \tau, \xi)| \geq |\tau|^m \left(1 - \frac{|\alpha_1(t, \xi)|}{|\tau|} - \cdots - \frac{|\alpha_m(t, \xi)|}{|\tau|^m} \right)$$

$$> 2M|\xi|^m \left(1 - \frac{1}{2} - \frac{1}{4M} - \cdots - \frac{1}{2^m M^{m-1}} \right) > 0,$$

hence $|\tau(t, \xi)| \leq 2M|\xi|$ for all $\xi \in \mathbb{R}^n$. Thus we establish (3.21).

Differentiating (3.19) with respect to t, we get

$$\frac{\partial L(t, \tau, \xi)}{\partial t} = \sum_{j=0}^m \partial_t \alpha_{m-j}(t, \xi)\tau^j = -\sum_{k=1}^m \partial_t \varphi_k(t, \xi) \prod_{r \neq k} (\tau - \varphi_r(t, \xi)).$$

Setting $\tau = \varphi_k(t, \xi)$, we obtain

$$(3.22) \qquad \partial_t \varphi_k(t, \xi) \prod_{r \neq k} (\varphi_k(t, \xi) - \varphi_r(t, \xi)) = -\sum_{j=0}^m \partial_t \alpha_{m-j}(t, \xi)\varphi_k(t, \xi)^j.$$

The positive homogeneity of order one of $\partial_t \varphi_k(t, \xi)$ is an immediate consequence of (3.22). Now, by using (3.20), (3.21), and the assumption that $|\xi|^{-j}\partial_t \alpha_j(\cdot, \xi) \in L^1(\mathbb{R}; L^\infty(\mathbb{R}^n \backslash 0))$ for all j, we conclude that $\partial_t \varphi_k(\cdot, \xi/|\xi|) \in L^1(\mathbb{R}; L^\infty(\mathbb{R}^n \backslash 0))$ for $k = 1, \ldots, m$. The proof is complete. $\square$

To state the main auxiliary result on the representation of solutions, we prepare the next lemma.

Lemma 3.2 (Mizohata [18] (Proposition 6.4)). *Assume* (3.18)–(3.20). *Then there exists a matrix* $\mathcal{N} = \mathcal{N}(t, \xi)$ *of homogeneous order of* 0 *in* ξ *satisfying the following properties*:
(i) $\mathcal{N}(t, \xi)A(t, \xi/|\xi|) = \mathcal{D}(t, \xi)\mathcal{N}(t, \xi)$, *where*

$$\mathcal{D}(t, \xi) = \mathrm{diag}\,(\varphi_1(t, \xi/|\xi|), \ldots, \varphi_m(t, \xi/|\xi|));$$

(ii) $\displaystyle\inf_{\xi \in \mathbb{R}^n \backslash 0, t \in \mathbb{R}} |\det \mathcal{N}(t, \xi))| > 0$;
(iii) $\mathcal{N}(t, \xi) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})$ *and* $\partial_t \mathcal{N}(t, \xi) \in L^1(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})$.

The next proposition is known as Levinson's lemma in the theory of ordinary differential equations (see Coddington and Levinson [3], and also Hartman [9]); the new feature here is the additional dependence on ξ, which is crucial for our analysis (see also Proposition 2.3 from [17] and Theorem 3.1 from [16]).

Proposition 3.3. *Assume* (3.18)–(3.20). *Let* $\mathcal{N}(t,\xi)$ *be the diagonaliser of* $A(t,\xi/|\xi|)$ *constructed in Lemma 3.2. Then there exist vector-valued functions* $\boldsymbol{a}^j(t,\xi)$, $j = 0,1,\ldots,m-1$, *determined by the initial value problem*

$$D_t\boldsymbol{a}^j(t,\xi) = C(t,\xi)\boldsymbol{a}^j(t,\xi), \qquad \left(\boldsymbol{a}^0(0,\xi),\cdots,\boldsymbol{a}^{m-1}(0,\xi)\right) = \mathcal{N}(0,\xi),$$

with $\quad C(t,\xi) = \Phi(t,\xi)^{-1}(D_t\mathcal{N}(t,\xi))\mathcal{N}(t,\xi)^{-1}\Phi(t,\xi) \in L^1(\mathbb{R};(L^\infty(\mathbb{R}^n\backslash 0))^{m^2})$,

such that the solution $U(t,x)$ *of* (3.17) *is represented by*

$$(3.23) \qquad U(t,x) = \sum_{j=0}^{m-1}\mathscr{F}^{-1}\left[\mathcal{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}^j(t,\xi)\widehat{f}_j(\xi)\right](x),$$

where $\mathscr{F}^{-1}$ *stands for the inverse Fourier transform and we put*

$$\Phi(t,\xi) = \mathrm{diag}\left(e^{i\int_0^t\varphi_1(\tau,\xi)\,d\tau},\cdots,e^{i\int_0^t\varphi_m(\tau,\xi)\,d\tau}\right).$$

Moreover, the following estimates hold:

$$(3.24) \qquad \sup_{t\in\mathbb{R}}\left\|\boldsymbol{a}^j(t,\xi)\right\|_{(L^\infty(\mathbb{R}^n\backslash 0))^m} \leq C$$

for $j = 0,1,\ldots,m-1$.

Proof. Applying the Fourier transform on $\mathbb{R}^n_x$, we get the following ordinary differential system from (3.17):

$$(3.25) \qquad D_t\boldsymbol{v} = A(t,\xi/|\xi|)|\xi|\boldsymbol{v}, \quad (\boldsymbol{v} = \widehat{U}).$$

Multiplying (3.25) by $\mathcal{N} = \mathcal{N}(t,\xi)$ from Lemma 3.2 and putting $\mathcal{N}\boldsymbol{v} = \boldsymbol{w}$, we get

$$(3.26) \qquad D_t\boldsymbol{w} = \mathscr{D}|\xi|\boldsymbol{w} + (D_t\mathcal{N})\boldsymbol{v} = \left(\mathscr{D}|\xi| + (D_t\mathcal{N})\mathcal{N}^{-1}\right)\boldsymbol{w},$$

since $\mathcal{N}A(t,\xi/|\xi|) = \mathscr{D}\mathcal{N}$ by Lemma 3.2. We can expect that the solutions of (3.26) are asymptotic to some solution of

$$(3.27) \qquad D_t\boldsymbol{y} = \mathscr{D}|\xi|\boldsymbol{y}.$$

Let $\Phi(t,\xi)$ be the fundamental matrix of (3.27), i.e.,

$$\Phi(t,\xi) = \mathrm{diag}\left(e^{i\int_0^t\varphi_1(\tau,\xi)\,d\tau},\cdots,e^{i\int_0^t\varphi_m(\tau,\xi)\,d\tau}\right).$$

If we perform the Wronskian transform $\boldsymbol{a}(t,\xi) = \Phi(t,\xi)^{-1}\boldsymbol{w}(t,\xi)$, then system (3.26) reduces to the system $D_t\boldsymbol{a} = C(t,\xi)\boldsymbol{a}$, where $C(t,\xi)$ is given by

$$C(t,\xi) = \Phi(t,\xi)^{-1}(D_t\mathcal{N}(t,\xi))\mathcal{N}(t,\xi)^{-1}\Phi(t,\xi).$$

We note that $C(t,\xi) \in L^1(\mathbb{R};(L^\infty(\mathbb{R}^n\backslash 0))^{m^2})$, since $D_t\mathcal{N}(t,\xi) \in L^1(\mathbb{R};(L^\infty(\mathbb{R}^n\backslash 0))^{m^2})$ by Lemma 3.2. Since $\boldsymbol{w}(t,\xi) = \Phi(t,\xi)\boldsymbol{a}(t,\xi)$ and $\mathcal{N}(t,\xi)\boldsymbol{v}(t,\xi) = \boldsymbol{w}(t,\xi)$, we get

$$\boldsymbol{v}(t,\xi) = \mathcal{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}(t,\xi).$$

Now let $(\boldsymbol{v}_0(t,\xi),\ldots,\boldsymbol{v}_{m-1}(t,\xi))$ be the fundamental matrix of (3.25). This means, in particular, that

$$(\boldsymbol{v}_0(0,\xi),\ldots,\boldsymbol{v}_{m-1}(0,\xi)) = I.$$

Then each $\boldsymbol{v}_j(t,\xi)$ can be represented by

$$\boldsymbol{v}_j(t,\xi) = \mathcal{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}^j(t,\xi),$$

where $\boldsymbol{a}^j(t,\xi)$ are the corresponding amplitude functions to $\boldsymbol{v}_j(t,\xi)$. Since $\widehat{U}(t,\xi) = \sum_{j=0}^{m-1} \boldsymbol{v}_j(t,\xi)\widehat{f}_j(\xi)$, we arrive at

$$\widehat{U}(t,\xi) = \sum_{j=0}^{m-1} \mathcal{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}^j(t,\xi)\widehat{f}_j(\xi).$$

Finally, let us find the estimates for the amplitude functions $\boldsymbol{a}^j(t,\xi)$. Recalling that $\boldsymbol{a}^j(t,\xi)$ satisfy the problem

$$D_t\boldsymbol{a}^j = C(t,\xi)\boldsymbol{a}^j \quad \text{with } (\boldsymbol{a}^0(0,\xi),\cdots,\boldsymbol{a}^{m-1}(0,\xi)) = \mathcal{N}(0,\xi),$$

we can write $\boldsymbol{a}^j(t,\xi)$ by the Picard series:

(3.28) $\boldsymbol{a}^j(t,\xi) =$

$$\left(I + i\int_0^t C(\tau_1,\xi)\,d\tau_1 + i^2\int_0^t C(\tau_1,\xi)\,d\tau_1 \int_0^{\tau_1} C(\tau_2,\xi)\,d\tau_2 + \cdots \right)\boldsymbol{a}^j(0,\xi).$$

This implies that

(3.29) $$\left|\boldsymbol{a}^j(t,\xi)\right| \le e^{c\int_{\mathbb{R}} \|\partial_\tau \mathcal{N}(\tau,\xi)\|_{L^\infty(\mathbb{R}^n)}\,d\tau}|\boldsymbol{a}^j(0,\xi)|,$$

where we have used the following fact:

Let $f(t)$ be a L^1_{loc}-function on $\mathbb{R}$. Then

$$e^{\int_s^t f(\tau)\,d\tau} = 1 + \int_s^t f(\tau_1)\,d\tau_1 + \int_s^t f(\tau_1)\,d\tau_1 \int_s^{\tau_1} f(\tau_2)\,d\tau_2 + \cdots.$$

The proof of Proposition 3.3 is now finished. $\qquad\square$

4. Proof of Theorem 2.1

In this section we shall prove the global well-posedness for Kirchhoff system (1.2). The strategy is to employ the Schauder-Tychonoff fixed point theorem. Let us consider the *linear* Cauchy problem (3.17) again:

$$\begin{cases} D_t U = A(t,D_x)U, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ U(0,x) = {}^T(f_0(x),f_1(x),\ldots,f_{m-1}(x)), \end{cases}$$

where $A(t,D_x)$ is the first order $m\times m$ pseudo-differential system, with symbol $A(t,\xi)$ positively homogeneous of order one. We assume that $A(t,\xi)$ satisfies the regularity condition (3.18) and the strictly hyperbolic condition (3.19)–(3.20). Notice that each characteristic root $\varphi_j(t,\xi)$ and its time derivative $\partial_t\varphi_j(t,\xi)$ are positively homogeneous of order one in ξ on account of Proposition 3.1. Furthermore, we observe from (3.23) and Plancherel's identity that if $U_0 \in \mathbb{H}^\sigma(\mathbb{R}^n)$ for some $\sigma \ge 0$, then the solution $U(t,x)$ to the linear equation (3.17) satisfies an energy estimate

(4.30) $$\|U(t,\cdot)\|_{\mathbb{H}^\sigma(\mathbb{R}^n)} \le C\|U_0\|_{\mathbb{H}^\sigma(\mathbb{R}^n)}, \quad \forall t \in \mathbb{R}.$$

Let us introduce a class of symbols of differential operators, which is convenient for the fixed point argument.

Class $\mathscr{K}$. *Given two constants $\Lambda > 0$ and $K > 0$, we say that a symbol $A(t, \xi)$ of a pseudo-differential operator $A(t, D_x)$ belongs to $\mathscr{K} = \mathscr{K}(\Lambda, K)$ if $A(t, \xi/|\xi|)$ belongs to $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})$ and satisfies*

$$\|A(t, \xi/|\xi|)\|_{L^\infty(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})} \leq \Lambda,$$

$$\int_{-\infty}^\infty \|\partial_t A(t, \xi/|\xi|)\|_{(L^\infty(\mathbb{R}^n \backslash 0))^{m^2}} \, dt \leq K.$$

The next lemma is the heart of our argument. It will be applied with a sufficiently small constant $K_0 > 0$ which will be fixed later, and for which all the constants in the estimates of the next lemma are positive.

Lemma 4.1. *Let $n \geq 1$. Assume that the symbol $A(t, \xi)$ of a pseudo-differential operator $A(t, D_x)$ satisfies (3.19)–(3.20) and belongs to $\mathscr{K} = \mathscr{K}(\Lambda, K)$ for some $\Lambda > 0$ and $0 < K \leq K_0$ with a sufficiently small constant $K_0 > 0$. Let $U \in C(\mathbb{R}; \mathbb{L}^2(\mathbb{R}^n))$ be a solution to the Cauchy problem*

$$D_t U = A(t, D_x)U, \quad U(0, x) = U_0(x) \in \mathbb{L}^2(\mathbb{R}^n) \cap \mathscr{Y}(\mathbb{R}^n),$$

and let $s(t)$ be the function

$$s(t) = \langle SU(t, \cdot), U(t, \cdot) \rangle_{\mathbb{L}^2(\mathbb{R}^n)}.$$

Then there exist two constants $M > 0$ and $c > 0$ independent of U and K such that

$$(4.31) \qquad \|A(s(t), \omega)\|_{(L^\infty(\mathbb{S}^{n-1}))^{m^2}}$$

$$\leq \|A(s(0), \omega)\|_{(L^\infty(\mathbb{S}^{n-1}))^{m^2}} + M\left(K\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)}^2 + \frac{1}{1 - cK}\|U_0\|_{\mathscr{Y}(\mathbb{R}^n)}\right),$$

$$(4.32) \quad \int_{-\infty}^\infty \|\partial_t \left[A(s(t), \omega)\right]\|_{(L^\infty(\mathbb{S}^{n-1}))^{m^2}} \, dt$$

$$\leq M\left(K\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)}^2 + \frac{1}{1 - cK}\|U_0\|_{\mathscr{Y}(\mathbb{R}^n)}\right).$$

We will be interested in sufficiently small $K_0 > 0$ so that we would have $1 - cK > 0$ in the estimates above.

Proof. The estimate (4.31) easily follows from (4.32) and the following identity:

$$A(s(t), \omega) = A(s(0), \omega) + \int_0^t \partial_\tau \left[A(s(\tau), \omega)\right] d\tau.$$

Hence it is sufficient to concentrate on proving (4.32). However, since

$$\|\partial_t \left[A(s(t), \omega)\right]\|_{(L^\infty(\mathbb{S}^{n-1}))^{m^2}} \leq C|s'(t)|,$$

we only have to show that

$$(4.33) \qquad \int_{-\infty}^\infty |s'(t)| \, dt \leq M\left(K\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)}^2 + \frac{1}{1 - cK}\|U_0\|_{\mathscr{Y}(\mathbb{R}^n)}\right).$$

We recall from Proposition 3.3 that

$$\widehat{U}(t,\xi) = \sum_{j=0}^{m-1} \mathcal{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}^j(t,\xi)\widehat{f}_j(\xi),$$

and its time-derivative version is given by

$$\widehat{U}'(t,\xi) = \sum_{j=0}^{m-1}\{\partial_t\mathcal{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}^j(t,\xi) + \mathcal{N}(t,\xi)^{-1}\partial_t\Phi(t,\xi)\boldsymbol{a}^j(t,\xi)$$

$$+ \mathcal{N}(t,\xi)^{-1}\Phi(t,\xi)\partial_t\boldsymbol{a}^j(t,\xi)\}\widehat{f}_j(\xi).$$

Plugging these equations into $s'(t)$, we can write

(4.34) $$s'(t) = 2\mathrm{Re}\left\langle S\widehat{U}'(t,\xi), \widehat{U}(t,\xi)\right\rangle_{\mathbb{L}^2(\mathbb{R}^n)} = 2\{I(t)+J(t)\},$$

where

$$I(t) =$$

$$\mathrm{Re}\sum_{j,k=0}^{m-1}\left\langle S\mathcal{N}(t,\xi)^{-1}\partial_t\Phi(t,\xi)\boldsymbol{a}^j(t,\xi)\widehat{f}_j(\xi), \mathcal{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}^k(t,\xi)\widehat{f}_k(\xi)\right\rangle_{\mathbb{L}^2(\mathbb{R}^n)},$$

$$J(t) =$$

$$\mathrm{Re}\sum_{j,k=0}^{m-1}\left\langle S\partial_t\mathcal{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}^j(t,\xi)\widehat{f}_j(\xi), \mathcal{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}^k(t,\xi)\widehat{f}_k(\xi)\right\rangle_{\mathbb{L}^2(\mathbb{R}^n)}$$

$$+ \left\langle S\mathcal{N}(t,\xi)^{-1}\Phi(t,\xi)\partial_t\boldsymbol{a}^j(t,\xi)\widehat{f}_j(\xi), \mathcal{N}(t,\xi)^{-1}\Phi(t,\xi)\boldsymbol{a}^k(t,\xi)\widehat{f}_k(\xi)\right\rangle_{\mathbb{L}^2(\mathbb{R}^n)}.$$

Since

$$\int_{-\infty}^{\infty}\left\{\left\|\partial_t\mathcal{N}(t,\xi)^{-1}\right\|_{(L^\infty(\mathbb{R}^n\backslash 0))^{m^2}} + \left\|\partial_t\boldsymbol{a}^j(t,\xi)\right\|_{(L^\infty(\mathbb{R}^n\backslash 0))^m}\right\}dt \leq CK$$

on account of Proposition 3.3, it follows that

(4.35) $$\int_{-\infty}^{\infty}|J(t)|\,dt \leq CK\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)}^2$$

with a certain constant $C > 0$.

It remains to estimate the oscillatory integral $I(t)$. Writing

$$\boldsymbol{a}^j(t,\xi) = {}^T(a^{1j}(t,\xi),\ldots,a^{mj}(t,\xi)) \text{ and } \mathcal{N}(t,\xi)^{-1} = (n^{lp}(t,\xi)),$$

we have

$$I(t) = \mathrm{Re}\sum_{j,k=0}^{m-1}\sum_{b,l,p,q=1}^{m} I_{j,k;b,l,p,q}(t),$$

where

$$I_{j,k;b,l,p,q}(t) = i\left\langle s_{bl}n^{lp}(t,\xi)\varphi_p(t,\xi)e^{i\vartheta_p(t,\xi)}a^{pj}(t,\xi)\widehat{f}_j, n^{bq}(t,\xi)e^{i\vartheta_q(t,\xi)}a^{qk}(t,\xi)\widehat{f}_k\right\rangle_{\mathbb{L}^2(\mathbb{R}^n)}$$

with

$$\vartheta_p(t,\xi) = \int_0^t \varphi_p(s,\xi)\,ds \quad (p = 1,\ldots,m).$$

Here the sum in $I_{j,k;b,l,p,q}(t)$ over $p = q$ does not contribute to $I(t)$. In fact, these integrals are pure imaginary. To see this fact, let us write

$$\varphi_p(t,\xi) = \varphi_p^+(t,\xi) - \varphi_p^-(t,\xi),$$

where $\varphi_p^+(t,\xi)$ and $\varphi_p^-(t,\xi)$ are positive and negative parts of $\varphi_p(t,\xi)$, respectively. Then we can write

$$i^{-1}I_{j,k;b,l,p,p}(t) =$$
$$\sum_{\pm} \pm \left\langle s_{bl}n^{lp}(t,\xi)\sqrt{\varphi_p^\pm(t,\xi)}a^{pj}(t,\xi)\widehat{f}_j, n^{bp}(t,\xi)\sqrt{\varphi_p^\pm(t,\xi)}a^{pk}(t,\xi)\widehat{f}_k \right\rangle_{L^2(\mathbb{R}^n)}$$

and since S is Hermitian, the sum $\sum_{j,k=0}^{m-1}\sum_{b,l,p=1}^{m} i^{-1}I_{j,k;b,l,p,p}(t)$ is real, and the real part

$$\mathrm{Re} \sum_{j,k=0}^{m-1} \sum_{b,l,p=1}^{m} I_{j,k;b,l,p,p}(t) = 0$$

vanishes. Therefore, by putting

$$\varphi_{pq}(t,\xi) = \sum_{b,l} s_{bl}n^{lp}(t,\xi)\varphi_p(t,\xi)\overline{n^{bq}(t,\xi)},$$

which are positively homogeneous of order one in ξ, we can write

$$I(t) = \mathrm{Re} \sum_{p\neq q}\sum_{j,k} \left\langle i\varphi_{pq}(t,\xi)e^{i\vartheta_p(t,\xi)}a^{pj}(t,\xi)\widehat{f}_j, e^{i\vartheta_q(t,\xi)}a^{qk}(t,\xi)\widehat{f}_k \right\rangle_{L^2(\mathbb{R}^n)}$$
$$= -\mathrm{Im} \sum_{p\neq q}\sum_{j,k} \left\langle \varphi_{pq}(t,\xi)e^{i\vartheta_p(t,\xi)}a^{pj}(t,\xi)\widehat{f}_j, e^{i\vartheta_q(t,\xi)}a^{qk}(t,\xi)\widehat{f}_k \right\rangle_{L^2(\mathbb{R}^n)}.$$

Now let us consider the functional

$$I_{p,q}(\eta(\cdot),t) = -\mathrm{Im} \sum_{j,k}\int_{\mathbb{S}^{n-1}} I_{p,q,j,k}(\eta(\omega),t)\,d\sigma(\omega),$$

where $\eta(\xi)$ is a function of homogeneous order zero, $d\sigma(\omega)$ is the $(n-1)$-dimensional Hausdorff measure, and we put

$$I_{p,q,j,k}(\eta(\omega),t) = \int_0^\infty e^{i\eta(\omega)\rho}a^{pj}(t,\rho\omega)\overline{a^{qk}(t,\rho\omega)}\varphi_{pq}(t,\omega)\widehat{f}_j(\rho\omega)\overline{\widehat{f}_k(\rho\omega)}\rho^n\,d\rho.$$

Furthermore, replacing $\eta(\cdot)$ in $I(\eta(\cdot),t)$ by a *real* parameter τ, we define

$$I_{p,q}^*(\tau) = \sup_{t\in\mathbb{R}}|I_{p,q}(\tau,t)|, \quad \tau\in\mathbb{R}.$$

If we prove that

$$(4.36) \qquad\qquad \sum_{p\neq q}\int_{-\infty}^\infty I_{p,q}^*(\tau)\,d\tau \leq \frac{C}{1-cK}|U_0|_{\mathscr{Y}(\mathbb{R}^n)},$$

then we conclude that

$$(4.37) \qquad \int_{-\infty}^{\infty} |I(t)| \, dt \leq \frac{C}{1 - cK} |U_0|_{\mathscr{Y}(\mathbb{R}^n)}.$$

Indeed, since

$$|I(t)| \leq C \sum_{p \neq q} \int_{\mathbb{S}^{n-1}} |I_{p,q}(\vartheta_p(t, \omega) - \vartheta_q(t, \omega), t)| \, d\sigma(\omega)$$

$$\leq C \sum_{p \neq q} \int_{\mathbb{S}^{n-1}} I_{p,q}^*(\vartheta_p(t, \omega) - \vartheta_q(t, \omega)) \, d\sigma(\omega),$$

it follows from the Fubini-Tonnelli theorem that

$$\int_{-\infty}^{\infty} |I(t)| \, dt \leq C \sum_{p \neq q} \int_{\mathbb{S}^{n-1}} \left(\int_{-\infty}^{\infty} I_{p,q}^*(\vartheta_p(t, \omega) - \vartheta_q(t, \omega)) \, dt \right) d\sigma(\omega).$$

Here we note that

$$\inf_{t \in \mathbb{R}, \omega \in \mathbb{S}^{n-1}} |\varphi_p(t, \omega) - \varphi_q(t, \omega)| \geq d(> 0) \qquad \text{for } p \neq q,$$

for some $d > 0$. Then, by changing the variable $\tau_\omega = \vartheta_{pq}(t, \omega) = \vartheta_p(t, \omega) - \vartheta_q(t, \omega)$, and by using (4.36), we can estimate

$$\sum_{p \neq q} \int_{\mathbb{S}^{n-1}} \left(\int_{-\infty}^{\infty} I_{p,q}^*(\vartheta_p(t, \omega) - \vartheta_q(t, \omega)) \, dt \right) d\sigma(\omega)$$

$$= \sum_{p \neq q} \int_{\mathbb{S}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{1}{\varphi_p(\vartheta_{pq}^{-1}(\tau_\omega, \omega)) - \varphi_q(\vartheta_{pq}^{-1}(\tau_\omega, \omega))} I_{p,q}^*(\tau_\omega) \, d\tau_\omega \right) d\sigma(\omega)$$

$$\leq d^{-1} \sum_{p \neq q} \int_{\mathbb{S}^{n-1}} \left(\int_{-\infty}^{\infty} I_{p,q}^*(\tau_\omega) \, d\tau_\omega \right) d\sigma(\omega)$$

$$\leq \frac{C}{1 - cK} |U_0|_{\mathscr{Y}(\mathbb{R}^n)},$$

which implies the estimate (4.37).

We now turn to prove the estimate (4.36). Recall the Picard series (3.28) for $a^{pj}(t, \xi)$; since $a^{pj}(0, \xi) = n_{pj}(0, \xi)$, it follows that

$$a^{pj}(t, \xi) = n_{pj}(0, \xi) + i \int_0^t c_{l_1}^p(\tau_1, \xi) n_{l_1 j}(0, \xi) \, d\tau_1$$

$$+ i^2 \int_0^t c_{l_1}^p(\tau_1, \xi) \, d\tau_1 \int_0^{\tau_1} c_{l_2}^{l_1}(\tau_2, \xi) n_{l_2 j}(0, \xi) \, d\tau_2 + \cdots,$$

where each entry $c_k^l(t, \xi)$ of $C(t, \xi)$ is of the form $n^{lq}(t, \xi) \partial_t n_{pk}(t, \xi) e^{i\vartheta_{pq}(t, \xi)}$, and the behaviour of $n^{lq}(t, \xi)$ is similar to that of $n_{pk}(t, \xi)$. In the sequel we omit the indices of $\vartheta_{pq}(t, \xi)$, $n_{pk}(t, \xi)$ and $n^{lq}(t, \xi)$. Then $\vartheta(t, \xi)$ is positively homogeneous of order one and $n(t, \xi)$ is homogeneous of order zero in ξ satisfying

$$(4.38) \qquad \int_{-\infty}^{\infty} \|\partial_t n(t, \cdot)\|_{L^\infty(\mathbb{R}^n \setminus 0)} \, dt \leq CK.$$

Plugging these series into $I_{p,q,j,k}(\tau,t)$, we extract the integrals depending on K^j for $j = 0, 1, 2, \ldots$, and estimate as follows:

(i) *Integrals independent of K* are easily handled by the estimate

$$\int_{-\infty}^{\infty} \left| \int_{\mathbb{R}^n} e^{i\tau|\xi|} n(t,\xi)^2 \varphi_{pq}(t,\xi) \widehat{f}_j(\xi) \overline{\widehat{f}_k(\xi)} \, d\xi \right| d\tau \leq C |U_0|_{\mathscr{Y}(\mathbb{R}^n)}.$$

Indeed, making the change of variable $\xi = \rho\omega$ ($\rho = |\xi|$, $\omega = \xi/|\xi| \in \mathbb{S}^{n-1}$), the left-hand side becomes

$$\int_{-\infty}^{\infty} \left| \int_{\mathbb{S}^{n-1}} n(t,\omega)^2 \varphi_{pq}(t,\omega) \left(\int_0^{\infty} e^{i\tau\rho} \widehat{f}_j(\rho\omega) \overline{\widehat{f}_k(\rho\omega)} \rho^n \, d\rho \right) d\sigma(\omega) \right| d\tau$$

$$\leq C \int_{-\infty}^{\infty} \left(\int_{\mathbb{S}^{n-1}} \left| \int_0^{\infty} e^{i\tau\rho} \widehat{f}_j(\rho\omega) \overline{\widehat{f}_k(\rho\omega)} \rho^n \, d\rho \right| d\sigma(\omega) \right) d\tau \leq C |U_0|_{\mathscr{Y}(\mathbb{R}^n)},$$

since $|n(t,\omega)^2 \varphi_{pq}(t,\omega)| \leq C$ for all $t \in \mathbb{R}$.

(ii) *Integrals depending on K* are reduced to the following:

$$\int_{-\infty}^{\infty} \left| \int_{\mathbb{R}^n} e^{i\tau|\xi|} i \left(\int_0^t \partial_{\tau_1} n(\tau_1,\xi) e^{i\vartheta(\tau_1,\xi)} \, d\tau_1 \right) n(0,\xi) \varphi_{pq}(t,\xi) \widehat{f}_j(\xi) \overline{\widehat{f}_k(\xi)} \, d\xi \right| d\tau.$$

Making the change of variable $\xi = \rho\omega$, using the bounds $|n(0,\omega)\varphi_{pq}(t,\omega)| \leq C$ and Fubini's theorem, we estimate the above integrals as

$$C \int_{-\infty}^{\infty} \int_{\mathbb{S}^{n-1}} \left(\int_0^t |\partial_{\tau_1} n(\tau_1,\omega)| \left| \int_0^{\infty} e^{i(\tau+\vartheta(\tau_1,\omega))\rho} \widehat{f}_j(\rho\omega) \overline{\widehat{f}_k(\rho\omega)} \rho^n \, d\rho \right| d\tau_1 \right) d\sigma(\omega) d\tau.$$

Since $U_0 \in \mathscr{Y}(\mathbb{R}^n)$, resorting to the invariance property of Lebesgue integrals with respect to the measure $d\tau$ and estimate (4.38), we conclude that the above integrals can be estimated as

$$C \int_0^t \|\partial_{\tau_1} n(\tau_1,\cdot)\|_{L^{\infty}(\mathbb{S}^{n-1})} \, d\tau_1 \times$$

$$\int_{-\infty}^{\infty} \left(\int_{\mathbb{S}^{n-1}} \left| \int_0^{\infty} e^{i\tau\rho} \widehat{f}_j(\rho\omega) \overline{\widehat{f}_k(\rho\omega)} \rho^n \, d\rho \right| d\sigma(\omega) \right) d\tau$$

$$\leq CK |U_0|_{\mathscr{Y}(\mathbb{R}^n)}.$$

(iii) *In the integrals depending on K^j for $j \geq 2$,* the factors

$$e^{i\tau\rho} \underbrace{\int \cdots \int}_{j} \partial_{\tau_1} n_1(\tau_1,\omega) e^{i\vartheta_1(\tau_1,\rho\omega)} \cdots \partial_{\tau_j} n_j(\tau_j,\omega) e^{i\vartheta_j(\tau_j,\rho\omega)} \, d\tau_1 \cdots d\tau_j$$

appear. Writing oscillatory factors as $e^{i(\tau+\vartheta(\tau_1,\omega)+\cdots+\vartheta(\tau_j,\omega))\rho}$, one can also handle such a factor by the invariance property of Lebesgue integrals. As a result, by using estimate (4.38), the present terms are bounded by $c^j K^j |U_0|_{\mathscr{Y}(\mathbb{R}^n)}$ for some constant $c > 0$.

Summing up these integrals and noting that $0 < K < 1$, we arrive at the estimate (4.36), namely at

$$\int_{-\infty}^{\infty} I^*(\tau)\, d\tau \le C(1 + cK + c^2 K^2 + \cdots)|U_0|_{\mathscr{Y}(\mathbb{R}^n)} = \frac{C}{1 - cK}|U_0|_{\mathscr{Y}(\mathbb{R}^n)},$$

provided that $0 < K \le K_0$ for some sufficiently small constant $K_0 > 0$. In conclusion, by combining (4.35) and (4.37), we get (4.33). The proof of Lemma 4.1 is now finished. $\qquad\square$

Proof of Theorem 2.1. We employ the Schauder-Tychonoff fixed point theorem. Let $A(t, \xi) \in \mathscr{K}$, and we fix the data $U_0 \in \mathbb{L}^2(\mathbb{R}^n) \cap \mathscr{Y}(\mathbb{R}^n)$. Then it follows from Lemma 4.1 that the mapping

$$\Theta : A(t, \xi) \mapsto A(s(t), \xi)$$

maps $\mathscr{K} = \mathscr{K}(\Lambda, K)$ into itself provided that $\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)}^2 + \|U_0\|_{\mathscr{Y}(\mathbb{R}^n)}$ is sufficiently small, with $\Lambda > 2\|A(0, \xi/|\xi|)\|_{(L^\infty(\mathbb{R}^n \backslash 0))^{m^2}}$ and sufficiently small $0 < K < K_0$. Now $\mathscr{K}$ may be regarded as the convex subset of the Fréchet space $L_{\mathrm{loc}}^\infty(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})$, and we endow $\mathscr{K}$ with the induced topology. We shall show that $\mathscr{K}$ is compact in $L_{\mathrm{loc}}^\infty(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})$ and the mapping Θ is continuous on $\mathscr{K}$.

Compactness of $\mathscr{K}$. Since $\mathscr{K}$ is uniformly bounded and equi-continuous on every compact t-interval, one can deduce from the Ascoli-Arzelà theorem that $\mathscr{K}$ is relatively compact in $L_{\mathrm{loc}}^\infty(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})$, and it is sequentially compact. This means that every sequence $\{A_j(t, \xi/|\xi|)\}_{j=1}^\infty$ in $\mathscr{K}$ has a subsequence, denoted by the same, converging to some $A(\cdot, \xi/|\xi|) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})$:

$$\begin{cases} A_j(t, \xi/|\xi|) \underset{(j \to \infty)}{\to} A(t, \xi/|\xi|) & \text{in } L_{\mathrm{loc}}^\infty(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2}), \\ \|A(t, \xi/|\xi|)\|_{L_{\mathrm{loc}}^\infty(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})} \le \Lambda, \end{cases}$$

where we used the fact that the absolute continuity of $\{A_j(t, \xi/|\xi|)\}$ is uniform in j on account of the Vitali-Hahn-Saks theorem (see e.g., §2 in Chapter II from [26]), since the finite limit $\lim_{j \to \infty} \int_s^t \partial_\tau A_j(\tau, \xi/|\xi|)\, d\tau$ exists for every interval (s, t). Moreover, the derivative $\partial_t A(t, \xi/|\xi|)$ exists almost everywhere on $\mathbb{R}$. Now, for the derivative $\partial_t A(t, \xi/|\xi|)$, if we prove that

$$(4.39) \qquad \int_{-\infty}^{+\infty} \|\partial_t A(t, \xi/|\xi|)\|_{(L^\infty(\mathbb{R}^n \backslash 0))^{m^2}}\, dt \le K,$$

then $A(t, \xi/|\xi|) \in \mathscr{K}$, which proves the compactness of $\mathscr{K}$.

For the proof of the estimate (4.39), we observe from Theorem 4 in §1 of Chapter V of [26] that the sequence $\{\partial_t A_j(\cdot, \xi/|\xi|)\}$ converges weakly to some matrix-valued function $B(\cdot, \xi/|\xi|) \in L^1(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})$ as $j \to \infty$, since the finite limit $\lim_{j \to \infty} \int_s^t \partial_\tau A_j(\tau, \xi/|\xi|)\, d\tau$ exists for every interval (s, t) and $\{\partial_t A_j(\cdot, \xi/|\xi|)\}$ is uniformly bounded in $L^1(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})$:

$$(4.40) \qquad \int_{-\infty}^{\infty} \|\partial_t A_j(t, \xi/|\xi|)\|_{(L^\infty(\mathbb{R}^n \backslash 0))^{m^2}}\, dt \le K.$$

By standard arguments we can conclude that $\partial_t A(t, \xi/|\xi|) = B(t, \xi/|\xi|)$ for a.e. $t \in \mathbb{R}$. Hence (4.39) is true, since

$$\int_{-\infty}^{+\infty} \|\partial_t A(t, \xi/|\xi|)\|_{(L^\infty(\mathbb{R}^n \backslash 0))^{m^2}} \, dt \leq \liminf_{j \to \infty} \int_{-\infty}^{+\infty} \|\partial_t A_j(t, \xi/|\xi|)\|_{(L^\infty(\mathbb{R}^n \backslash 0))^{m^2}} \, dt \leq K,$$

where we used (4.40).

Continuity of Θ on $\mathscr{K}$. Let us take a sequence $\{A_k(t, \xi/|\xi|)\}$ in $\mathscr{K}$ such that

(4.41) $A_k(t, \xi/|\xi|) \to A(t, \xi/|\xi|) \in \mathscr{K}$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})$ $(k \to \infty)$,

and let $U_k(t, x)$ and $U(t, x)$ be the corresponding solutions to $A_k(t, \xi)$ and $A(t, \xi)$, respectively, with fixed data U_0 satisfying the assumption of Theorem 2.1. Put

(4.42) $s_k(t) = \langle SU_k(t), U_k(t) \rangle_{\mathbb{L}^2(\mathbb{R}^n)}.$

Then we prove that the images

$$A_k(s_k(t), \xi) = \Theta(A_k(t, \xi)) \text{ and } A(s(t), \xi) = \Theta(A(t, \xi))$$

satisfy

(4.43) $A_k(s_k(t), \xi/|\xi|) \to A(s(t), \xi/|\xi|)$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})$ $(k \to \infty)$.

Using Proposition 3.3 again, we can write

$$U(t, x) = \sum_{j=0}^{m-1} \mathscr{F}^{-1} \left[\mathscr{N}(t, \xi)^{-1} \Phi(t, \xi) \boldsymbol{a}^j(t, \xi) \widehat{f_j}(\xi) \right](x),$$

$$U_k(t, x) = \sum_{j=0}^{m-1} \mathscr{F}^{-1} \left[\mathscr{N}_k(t, \xi)^{-1} \Phi_k(t, \xi) \boldsymbol{a}_k^j(t, \xi) \widehat{f_j}(\xi) \right](x).$$

Notice that

(4.44) $\Phi_k(t, \xi) \to \Phi(t, \xi)$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})$ $(k \to \infty)$,

(4.45) $\mathscr{N}_k(t, \xi)^{-1} \to \mathscr{N}(t, \xi)^{-1}$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})$ $(k \to \infty)$

on account of (4.41). Furthermore, we have

(4.46) $\boldsymbol{a}_k^j(t, \xi) \to \boldsymbol{a}^j(t, \xi)$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^m)$ $(k \to \infty)$.

Indeed, we observe from previous argument that $\{\partial_t A_k(t, \xi/|\xi|)\}$ is weakly convergent to $\partial_t A(t, \xi/|\xi|)$ in $L^1(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})$, and hence, $\{\partial_t \mathscr{N}_k(t, \xi)\}$ is also weakly convergent to $\partial_t \mathscr{N}(t, \xi/|\xi|)$ in $L^1(\mathbb{R}; (L^\infty(\mathbb{R}^n \backslash 0))^{m^2})$. Thus we find from this observation and the Picard series (3.28) for $\boldsymbol{a}_k^j(t, \xi)$ that the convergence (4.46) is proved. Then, by using the Lebesgue dominated convergence theorem, we conclude from (4.42)–(4.46) that $s_k(t) \to s(t)$ $(k \to \infty)$, which implies (4.43).

Completion of the proof of Theorem 2.1. By using the Schauder–Tychonoff fixed point theorem, we can show that Θ has a fixed point in $\mathscr{K}$, with $K_0 > 0$ in Lemma 4.1 sufficiently small, so that constants in (4.31)-(4.32) are positive. Hence, we conclude that if $U_0 \in \mathbb{L}^2(\mathbb{R}^n) \cap \mathscr{Y}(\mathbb{R}^n)$, then the solutions $U(t, x)$ of

$$D_t U = A(t, D_x) U$$

with the Cauchy data $U(0, x) = U_0(x)$ are solutions to the nonlinear system (1.2) and belong to $C(\mathbb{R}; \mathbb{L}^2(\mathbb{R}^n))$. Furthermore, these solutions U satisfy the energy estimates (4.30).

Finally, we prove the uniqueness. Let U, V be two solutions to the nonlinear system (1.2) with $U(0, x) = V(0, x) = U_0(x)$, and let

$$s_U(t) = \langle SU(t), U(t)\rangle_{\mathbb{L}^2(\mathbb{R}^n)} \quad \text{and} \quad s_V(t) = \langle SV(t), V(t)\rangle_{\mathbb{L}^2(\mathbb{R}^n)}$$

be the corresponding nonlocal terms, respectively. In this time, we need to assume that $U_0 \in \mathbb{H}^1(\mathbb{R}^n)$. We observe from Proposition 3.3 and the property of the mapping Θ in the fixed point argument that, when we consider the integral representations of U and V, the functions $\Phi(t, \xi)$ and $\mathcal{N}(t, \xi)$ in the representation (3.23) may be replaced by

$$\Phi(s(t), \xi) = \mathrm{diag}\left(e^{i\int_0^t \varphi_1(s(\tau),\xi)\,d\tau}, \cdots, e^{i\int_0^t \varphi_m(s(\tau),\xi)\,d\tau}\right) \text{ and } \mathcal{N}(s(t), \xi),$$

respectively, where $s(t) = s_U(t)$ or $s_V(t)$. Since $\boldsymbol{a}^j(t, \xi)$ are solutions to the linear system $D_t\boldsymbol{a}^j(t, \xi) = C(t, \xi)\boldsymbol{a}^j(t, \xi)$, where

$$C(t, \xi) = \Phi(t, \xi)^{-1}(D_t\mathcal{N}(t, \xi))\mathcal{N}(t, \xi)^{-1}\Phi(t, \xi),$$

the amplitude functions $\boldsymbol{a}^j(s, \xi)$ for nonlinear system satisfy ordinary differential systems

$$(4.47) \qquad\qquad D_t\boldsymbol{a}^j(s(t), \xi) = C(s(t), \xi)\boldsymbol{a}^j(s(t), \xi),$$

where

$$C(s(t), \xi) = \Phi(s(t), \xi)^{-1}(D_t\mathcal{N}(s(t), \xi))\mathcal{N}(s(t), \xi)^{-1}\Phi(s(t), \xi)$$

are in $L^1(\mathbb{R}; (L^\infty(\mathbb{R}^n\backslash 0))^{m^2})$. Thus the solutions U, V of (1.2) have the following forms:

$$U(t, x) = \sum_{j=0}^{m-1} \mathscr{F}^{-1}\left[\mathcal{N}(s_U(t), \xi)^{-1}\Phi(s_U(t), \xi)\boldsymbol{a}^j(s_U(t), \xi)\widehat{f_j}(\xi)\right](x),$$

$$V(t, x) = \sum_{j=0}^{m-1} \mathscr{F}^{-1}\left[\mathcal{N}(s_V(t), \xi)^{-1}\Phi(s_V(t), \xi)\boldsymbol{a}^j(s_V(t), \xi)\widehat{f_j}(\xi)\right](x).$$

Then we can write

(4.48)

$$\|U(t) - V(t)\|_{\mathbb{L}^2(\mathbb{R}^n)}^2$$

$$= \sum_{j,k=0}^{m-1} \int_{\mathbb{R}^n} \left(\boldsymbol{b}^j(s_U(t), \xi) - \boldsymbol{b}^j(s_V(t), \xi)\right) \cdot \overline{\left(\boldsymbol{b}^k(s_U(t), \xi) - \boldsymbol{b}^k(s_V(t), \xi)\right)}\widehat{f_j}(\xi)\overline{\widehat{f_k}(\xi)}\,d\xi,$$

where we put

$$\boldsymbol{b}^j(s_U(t), \xi) = \mathcal{N}(s_U(t), \xi)^{-1}\Phi(s_U(t), \xi)\boldsymbol{a}^j(s_U(t), \xi),$$

$$\boldsymbol{b}^j(s_V(t), \xi)) = \mathcal{N}(s_V(t), \xi)^{-1}\Phi(s_V(t), \xi)\boldsymbol{a}^j(s_V(t), \xi).$$

The functional $s_U(t)$ is Lipschitz with respect to U since

$$|s_U(t) - s_V(t)| \leq |\langle S(U(t) - V(t)), U(t)\rangle_{\mathbb{L}^2(\mathbb{R}^n)}| + |\langle SV(t), U(t) - V(t)\rangle_{\mathbb{L}^2(\mathbb{R}^n)}|$$

(4.49)
$$\leq C\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)}\|U(t) - V(t)\|_{\mathbb{L}^2(\mathbb{R}^n)}.$$

Since $A(s, \xi/|\xi|)$ is Lipschitz with respect to s, $\mathcal{N}(s,\xi)^{-1}$ and $\varphi_k(s,\xi)$ also depend on s Lipschitz continuously; thus we find from (4.49) that

(4.50)
$$\left\|\mathcal{N}(s_U(t),\xi)^{-1} - \mathcal{N}(s_V(t),\xi)^{-1}\right\|_{(L^\infty(\mathbb{R}^n\setminus 0))^{m^2}}$$
$$\leq C|s_U(t) - s_V(t)| \leq C\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)}\|U(t) - V(t)\|_{\mathbb{L}^2(\mathbb{R}^n)},$$

(4.51)
$$\left\|\Phi(s_U(t),\xi) - \Phi(s_V(t),\xi)\right\| \leq \sum_{k=1}^m \left| e^{i\int_0^t \varphi_k(s_U(\tau),\xi)\,d\tau} - e^{i\int_0^t \varphi_k(s_V(\tau),\xi)\,d\tau} \right|$$

$$\leq \sum_{k=1}^m \int_0^t |\varphi_k(s_U(\tau),\xi) - \varphi_k(s_V(\tau),\xi)|\,d\tau$$

$$\leq C|\xi| \int_0^t |s_U(\tau) - s_V(\tau)|\,d\tau$$

$$\leq C|\xi|\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)} \int_0^t \|U(\tau) - V(\tau)\|_{\mathbb{L}^2(\mathbb{R}^n)}\,d\tau,$$

where $\|\cdot\|$ denotes a matrix norm. Furthermore, the amplitude functions $\boldsymbol{a}^j(s,\xi)$ satisfy the following estimates:

(4.52)
$$\|\boldsymbol{a}^j(s_U(t),\xi) - \boldsymbol{a}^j(s_V(t),\xi)\|_{(L^\infty(\mathbb{R}^n\setminus 0))^m} \leq C\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)}\|U(t) - V(t)\|_{\mathbb{L}^2(\mathbb{R}^n)}.$$

In fact, since $\boldsymbol{a}^j(s,\xi)$ satisfy the ordinary differential system (4.47) with $C(s,\xi) \in L^1((0,\delta); (L^\infty(\mathbb{R}^n\setminus 0))^{m^2})$, it follows that

$$D_s\boldsymbol{a}^j(s,\xi) = C(s,\xi)\boldsymbol{a}^j(s,\xi),$$

and hence, $\boldsymbol{a}^j(s,\xi)$ are Lipschitz in s. Therefore, there exists a constant $L > 0$ such that

$$\|\boldsymbol{a}^j(s_U(t),\xi) - \boldsymbol{a}^j(s_V(t),\xi)\|_{(L^\infty(\mathbb{R}^n\setminus 0))^m}$$
$$\leq L|s_U(t) - s_V(t)|$$
$$\leq C\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)}\|U(t) - V(t)\|_{\mathbb{L}^2(\mathbb{R}^n)},$$

where we used (4.49) in the last step. This proves (4.52). Summarising (4.50)–(4.52), we conclude that

$$|\boldsymbol{b}^j(s_U(t),\xi) - \boldsymbol{b}^j(s_V(t),\xi)|$$
$$\leq C\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)}\left(\|U(t) - V(t)\|_{\mathbb{L}^2(\mathbb{R}^n)} + |\xi|\int_0^t \|U(\tau) - V(\tau)\|_{\mathbb{L}^2(\mathbb{R}^n)}\,d\tau\right).$$

Thus, (4.48) together with these estimates imply that

$$\|U(t) - V(t)\|^2_{\mathbb{L}^2(\mathbb{R}^n)} \le C\Bigg\{ \|U_0\|^4_{\mathbb{L}^2(\mathbb{R}^n)}\|U(t) - V(t)\|^2_{\mathbb{L}^2(\mathbb{R}^n)}$$

$$+ \|U_0\|^2_{\mathbb{L}^2(\mathbb{R}^n)}\|D_x U_0\|^2_{\mathbb{L}^2(\mathbb{R}^n)}\left(\int_0^t \|U(\tau) - V(\tau)\|_{\mathbb{L}^2(\mathbb{R}^n)}\, d\tau \right)^2 \Bigg\}.$$

Since $\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)}$ is sufficiently small, we obtain

$$\|U(t) - V(t)\|_{\mathbb{L}^2(\mathbb{R}^n)} \le C(\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)})\|D_x U_0\|_{\mathbb{L}^2(\mathbb{R}^n)} \int_0^t \|U(\tau) - V(\tau)\|_{\mathbb{L}^2(\mathbb{R}^n)}\, d\tau$$

for some function $C(\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)})$. Thus, applying Gronwall's lemma to the above inequality, we conclude that $U(t) = V(t)$ for all $t \in \mathbb{R}$. This proves the uniqueness of solutions. The proof of Theorem 2.1 is now finished. $\square$

5. A FINAL REMARK

Observing the inclusion (2.6), we can also prove:

Theorem 5.1. *Let $n \ge 1$ and $\varkappa \in (1, n+1]$. Assume that system (1.2) is strictly hyperbolic, and that $A(s, \xi) = (a_{jk}(s, \xi))_{j,k=1}^m$ is an $m \times m$ matrix, positively homogeneous of order one in ξ, whose entries $a_{jk}(s, \xi)$ satisfy $|\xi|^{-1+|\alpha|}\partial_\xi^\alpha a_{jk}(s, \xi) \in \mathrm{Lip}([0, \delta]; L^\infty(\mathbb{R}^n\backslash 0))$ for any $0 \le |\alpha| \le [\varkappa] + 1$ and for some $\delta > 0$. If U_0 are small in the space $\mathbb{H}^1_\varkappa(\mathbb{R}^n)$, then system (1.2) has a unique solution $U \in C(\mathbb{R}; \mathbb{H}^1(\mathbb{R}^n)) \cap C^1(\mathbb{R}; \mathbb{L}^2(\mathbb{R}^n))$.*

Outline of the proof. In order to prove the theorem, let us introduce a subclass of $\mathscr{K}$ as follows:

Class $\mathscr{K}'$. *Given three constants $\Lambda > 0$, $K > 0$ and $\varkappa > 1$, we say that the symbol $A(t, \xi)$ of a pseudo-differential operator $A(t, D_x)$ belongs to $\mathscr{K}' = \mathscr{K}'(\Lambda, K, \varkappa)$ if $A(t, \xi)$ belongs to $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}; (C^{[\varkappa]+1}(\mathbb{R}^n\backslash 0))^{m^2})$ and satisfies*

$$\|A(t, \xi/|\xi|)\|_{L^\infty(\mathbb{R};(L^\infty(\mathbb{R}^n\backslash 0))^{m^2})} \le \Lambda,$$

$$\left\| |\xi|^{-1+|\alpha|}\partial_\xi^\alpha \partial_t A(t, \xi) \right\|_{(L^\infty(\mathbb{R}^n\backslash 0))^{m^2}} \le C_\alpha K \langle t \rangle^{-\varkappa}, \quad 0 \le \forall |\alpha| \le [\varkappa] + 1.$$

We have the following lemma:

Lemma 5.2. *Let $n \ge 1$ and $1 < \varkappa \le n + 1$. Assume that the symbol $A(t, \xi)$ of a differential operator $A(t, D_x)$ satisfies (3.19)–(3.20) and belongs to $\mathscr{K}'$ for some $\Lambda > 0$ and $0 < K < K_0$ with sufficiently small K_0. Let $U \in C(\mathbb{R}; \mathbb{H}^1(\mathbb{R}^n)) \cap C^1(\mathbb{R}; \mathbb{L}^2(\mathbb{R}^n))$ be a solution to the Cauchy problem*

$$D_t U = A(t, D_x)U, \quad U(0, x) = U_0(x) \in \mathbb{H}^1_\varkappa(\mathbb{R}^n),$$

and let $s(t)$ be the functional

$$s(t) = \langle SU(t, \cdot), U(t, \cdot)\rangle_{\mathbb{L}^2(\mathbb{R}^n)}.$$

Then, for any $0 \leq |\alpha| \leq [\varkappa] + 1$, there exist constants $M_\alpha > 0$ and $c_{\alpha,\varkappa} > 0$ independent of U such that

$$\|A(s(t), \xi/|\xi|)\|_{(L^\infty(\mathbb{R}^n \backslash 0))^{m^2}}$$

$$\leq \|A(s(0), \xi/|\xi|)\|_{(L^\infty(\mathbb{R}^n \backslash 0))^{m^2}} + M_0 \left(K\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)}^2 + \frac{1}{1 - c_{\alpha,0}K} \|U_0\|_{\mathbb{H}_\varkappa^1(\mathbb{R}^n)}^2 \right),$$

$$\left\| |\xi|^{-1+|\alpha|} \partial_\xi^\alpha \partial_t A(s(t), \xi) \right\|_{(L^\infty(\mathbb{R}^n \backslash 0))^{m^2}}$$

$$\leq M_\alpha \left(K\|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)}^2 + \frac{1}{1 - c_{\alpha,\varkappa}K} \|U_0\|_{\mathbb{H}_\varkappa^1(\mathbb{R}^n)}^2 \right) \langle t \rangle^{-\varkappa}.$$

The proof of Lemma 5.2 can be done by some modifications of the argument of Lemma 4.1, and we take $K_0 > 0$ small enough so that $1 - c_{\alpha,\varkappa}K_0 > 0$ for all α and $\varkappa$. The following lemma can be obtained by a similar proof as Lemma A.1 of D'Ancona & Spagnolo [5] (see also Lemma 3.2 of [14], and [23]).

Lemma 5.3. *Let $n \geq 1$ and $\varkappa \in (1, n+1]$. Assume that $\varphi(\xi) \in C(\mathbb{R}^n \backslash 0)$ is a positively homogeneous function of order one. Then*

$$\int_{\mathbb{S}^{n-1}} \left| \int_0^\infty e^{i\tau\rho} \hat{f}_1(\rho\omega) \hat{f}_2(\rho\omega) \varphi(\omega) \rho^n \, d\rho \right| d\sigma(\omega) \leq C_\varkappa \langle \tau \rangle^{-\varkappa} \|f_1\|_{H_\varkappa^1(\mathbb{R}^n)} \|f_2\|_{H_\varkappa^1(\mathbb{R}^n)}$$

for any $f_1, f_2 \in \mathscr{S}(\mathbb{R}^n)$, where τ is a real parameter.

In particular, observing the proof of Lemma 3.2 (see Proposition 6.4 in [18]), one can check that if $A(t, \xi) \in \mathscr{K}'$, then derivatives of $\mathscr{N}(t, \xi) = (n_{jk}(t, \xi))_{j,k=1}^m$ and $\mathscr{N}(t, \xi)^{-1} = (n^{pq}(t, \xi))_{p,q=1}^m$ satisfy

$$\left\| |\xi|^{-|\alpha|} \partial_\xi^\alpha n_{jk}(t, \xi) \right\|_{L^\infty(\mathbb{R}^n \backslash 0)}, \; \left\| |\xi|^{-|\alpha|} \partial_\xi^\alpha n^{pq}(t, \xi) \right\|_{L^\infty(\mathbb{R}^n \backslash 0)} \leq C_\alpha \Lambda,$$

$$\left\| |\xi|^{-|\alpha|} \partial_\xi^\alpha \partial_t n_{jk}(t, \xi) \right\|_{L^\infty(\mathbb{R}^n \backslash 0)}, \; \left\| |\xi|^{-|\alpha|} \partial_\xi^\alpha \partial_t n^{pq}(t, \xi) \right\|_{L^\infty(\mathbb{R}^n \backslash 0)} \leq C_\alpha K \langle t \rangle^{-\varkappa},$$

for any $0 \leq |\alpha| \leq [\varkappa] + 1$. Furthermore, t-derivatives of amplitudes $\boldsymbol{a}^j(t, \xi)$ are estimated by

$$\|\partial_t \boldsymbol{a}^j(t, \xi)\|_{(L^\infty(\mathbb{R}^n \backslash 0))^m} \leq CK \langle t \rangle^{-\varkappa}.$$

Combining these estimates and the decay estimates for oscillatory integrals given in Lemma 5.3, we can perform the integration by parts with respect to $\rho = |\xi|$ in oscillatory integrals $I_{p,q}(\tau, t)$. Thus we find that

$$\sum_{p \neq q} I_{p,q}^*(\tau) \leq \frac{C_\varkappa}{1 - c_{\alpha,\varkappa}K} \|U_0\|_{\mathbb{H}_\varkappa^1(\mathbb{R}^n)}^2 \langle \tau \rangle^{-\varkappa}$$

for any $\varkappa \in (1, n+1]$, which implies that

$$|I(t)| \leq \frac{C_\varkappa}{1 - c_{\alpha,\varkappa}K} \|U_0\|_{\mathbb{H}_\varkappa^1(\mathbb{R}^n)}^2 \langle t \rangle^{-\varkappa}.$$

As to $J(t)$, we easily obtain

$$|J(t)| \leq CK \|U_0\|_{\mathbb{L}^2(\mathbb{R}^n)}^2 \langle t \rangle^{-\varkappa}.$$

Hence Lemma 5.2 is proved by combining the decay estimates for $I(t)$ and $J(t)$.

Resorting to Lemma 5.2, we can perform the fixed point argument as in the previous section (see also [14]), and as a result, the solution $U(t, x)$ to the linear system will be, of course, a solution to the original nonlinear system, which allows us to conclude the proof of Theorem 5.1. □

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